Networks: Spring 2009	Homework 1
David Easley and Jon Kleinberg	Due in Class February 4, 2009

Homework will be due at the start of class on the due date. We cannot accept late homework except for University-approved excuses (which include illness with a note from Gannett, a family emergency, or travel as part of a University sports team or other University activity).

Parts A and B: Please hand in the solutions to the different parts separately, to facilitate grading.

Reading: The questions below are primarily based on the material in Chapters 2-4 of the Networks book draft.

Part A

(1) In the social network depicted in Figure 1, with each edge labeled as either a strong or weak tie, which nodes satisfy the Strong Triadic Closure Property from Chapter 3, and which do not? Provide an explanation for your answer.

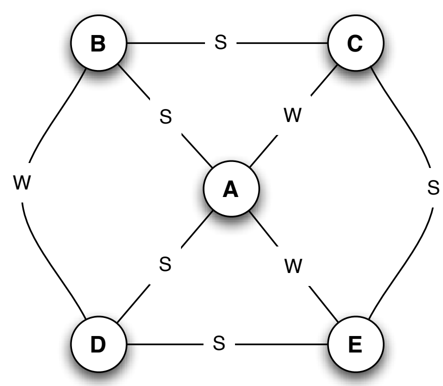


Figure 1: The graph for Question 1.

Some Questions on Joining a Network so as to Preserve Balance. When we think about structural balance, we can ask what happens when a new node tries to join a network in which there is existing friendship and hostility. In Figures 2–5, each pair of nodes is either friendly or hostile, as indicated by the + or − label on each edge.

First, consider the 3-node social network in Figure 2, in which all pairs of nodes know each other, and all pairs of nodes are friendly toward each other. Now, a fourth node D wants

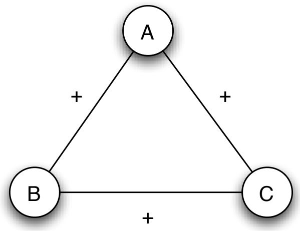
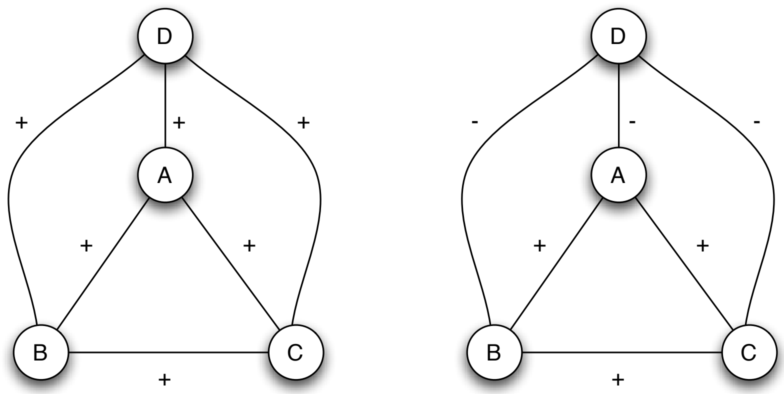


Figure 2: A 3-node social network in which all pairs of nodes know each other, and all pairs of nodes are friendly toward each other.



(a) *D joins the network by becoming friends with all nodes.*

(b) *D joins the network by becoming enemies with all nodes.*

Figure 3: There are two distinct ways in which node D can join the social network from Figure 2 without becoming involved in any unbalanced triangles.

to join this network, and establish either positive or negative relations with each existing node A, B, and C. It wants to do this in such a way that it doesn't become involved in any unbalanced triangles. (I.e. so that after adding D and the labeled edges from D, there are no unbalanced triangles that contain D.) Is this possible?

In fact, in this example, there are two ways for D to accomplish this, as indicated in Figure 3. First, D can become friends with all existing nodes; in this way, all the triangles containing it have three positive edges, and so are balanced. Alternately, it can become enemies with all existing nodes; in this way, each triangle containing it has exactly one positive edge, and again these triangles would be balanced.

So for this network, it was possible for D to join without becoming involved in any unbalanced triangles. However, the same is not necessarily possible for other networks.

We now consider this kind of question for some other networks.

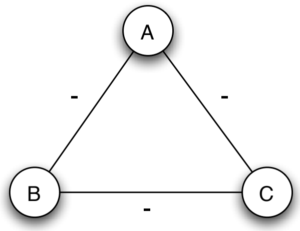


Figure 4: The graph for Question 2. All three nodes are mutual enemies.

(2) Consider the 3-node social network in Figure 4, in which all pairs of nodes know each other, and each pair is either friendly or hostile as indicated by the + or − label on each edge. A fourth node D wants to join this network, and establish either positive or negative relations with each existing node A, B, and C. Can node D do this in such a way that it doesn't become involved in any unbalanced triangles?

- If there is a way for D to do this, say how many different such ways there are, and give an explanation. (That is, how many different possible labelings of the edges out of D have the property that all triangles containing D are balanced?)
- If there is no such way for D to do this, give an explanation why not.

(In this and the subsequent questions, it possible to work out an answer by reasoning about the new node's options without having to check all possibilities.)

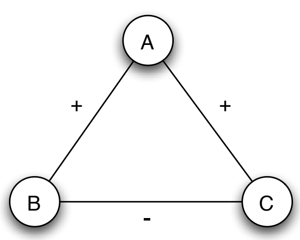


Figure 5: The graph for Question 3. Node A is friends with nodes B and C, who are enemies with each other.

(3) Same question, but for a different network. Consider the 3-node social network in Figure 5, in which all pairs of nodes know each other, and each pair is either friendly or hostile as indicated by the + or − label on each edge. A fourth node D wants to join this network, and establish either positive or negative relations with each existing node A, B, and C. Can node D do this in such a way that it doesn't become involved in any unbalanced triangles?

- If there is a way for D to do this, say how many different such ways there are, and give an explanation. (That is, how many different possible labelings of the edges out of D have the property that all triangles containing D are balanced?)
- If there is no such way for D to do this, give an explanation why not.

---

(4) Using what you've worked out in Questions 2 and 3, consider the following question. Take *any* labeled complete graph — on any number of nodes — that is not balanced; i.e. it contains at least one unbalanced triangle. (Recall that a labeled complete graph is a graph in which there is an edge between each pair of nodes, and each edge is labeled with either $+$ or $-$.) A new node X wants to join this network, by attaching to each node using a positive or negative edge. When, if ever, is it possible for X to do this in such a way that it does not become involved in any unbalanced triangles? Give an explanation for your answer. *(Hint: Think about any unbalanced triangle in the network, and how X must attach to the nodes in it.)*

Part B

One reason for graph theory's power as a modeling tool is the fluidity with which one can formalize properties of large systems using the language of graphs, and then systematically explore their consequences. In the remaining problems below, we will work through an example of this process using the concept of a *pivotal* node.

First, recall from Chapter 2 that a *shortest path* between two nodes is a path of the minimum possible length. We say that a node X is *pivotal* for a pair of distinct nodes Y and Z if X lies on every shortest path between Y and Z (and X is not equal to either Y or Z).

For example, in the graph in Figure 6, node B is pivotal for two pairs: the pair consisting of A and C, and the pair consisting of A and D. (Notice that B is not pivotal for the pair consisting of D and E since there are two different shortest paths connecting D and E, one of which (using C and F) doesn't pass through B. So B is not on *every* shortest path between D and E.) On the other hand, node D is not pivotal for any pairs.

---

(5) Give an example of a graph in which *every* node is pivotal for at least one pair of nodes. Explain your answer.

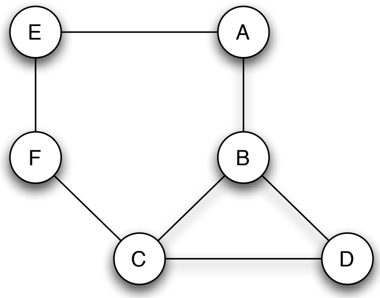


Figure 6: In this example, node B is pivotal for two pairs: the pair consisting of A and C, and the pair consisting of A and D. On the other hand, node D is not pivotal for any pairs.

(6) Give an example of a graph in which *every* node is pivotal for at least two different pairs of nodes. Explain your answer.

(7) Give an example of a graph having at least four nodes in which there is a single node X that is pivotal for *every* pair of nodes (not counting pairs that include X). Explain your answer.

(8) Finally, an open-ended question. In a paragraph, describe any relationships you see between the definition of a pivotal node considered here, and any of the notions that we've encountered in Chapters 2-4. These relationships can either be qualitative (i.e. that different definitions are trying to model aspects of a similar idea), or more formal (i.e. that there are precise relationships between certain of the definitions in Chapters 2-4 and the definition of a pivotal node). There is no single "right" answer to this question; rather, it's a chance to discuss the concepts in the past few questions in light of the notions from Chapters 2-4.