Networks: Spring 2009 **Homework 2**
David Easley and Jon Kleinberg **Due in Class February 13, 2008**

Homework will be due at the start of class on the due date. We cannot accept late homework except for University-approved excuses (which include illness with a note from Gannett, a family emergency, or travel as part of a University sports team or other University activity).

Parts A and B: Please hand in the different parts separately, to facilitate grading.

Reading: The questions below are primarily based on the material in Chapters 5 and 7 of the Networks book draft.

Part A

(1) In this question we will consider several two-player games. In each payoff matrix below the rows correspond to player A's strategies and the columns correspond to player B's strategies. The first entry in each box is player A's payoff and the second entry is player B's payoff.

(a) Find all Nash equilibria for the game described by the payoff matrix below.

		Player B	
		L	R
Player A	U	$1, 1$	$3, 2$
	D	$0, 3$	$4, 4$

(b) Find all Nash equilibria for the game described by the payoff matrix below (include an explanation for your answer).

		Player B	
		L	R
Player A	U	$5, 6$	$0, 10$
	D	$4, 4$	$2, 2$

[Hint: This game has a mixed strategy equilibrium. To find the equilibrium let the probability that player A uses strategy U be p and the probability that player B uses strategy L be q. As we learned in our analysis of matching pennies, if a player uses a mixed strategy (one that is not really just some pure strategy played with probability one) then the player must be indifferent between two pure strategies. That is, the strategies must have equal expected payoffs. So, for example, if p is not 0 or 1 then it must be the case that $5q+0(1-q) = 4q+2(1-q)$ as these are the expected payoffs to player A from U and D when player B uses probability q.]

(2) In class we defined the concept of Nash equilibrium for games with many strategies for each player, but all of our examples had only two strategies for each player. In this question we will consider a two-player game in which each player has three strategies.

Player B

		L	M	R
	U	$1,1$	$2,3$	$1,6$
Player A	M	$3,4$	$5,5$	$2,2$
	D	$1,10$	$4,7$	$0,4$

Find all the (pure strategy) Nash equilibria for this game.

(3) Consider the two-player game described by the payoff matrix below.

Player B

		L	R
Player A	U	$1,1$	$0,0$
	D	$0,0$	$4,4$

(a) Find all pure-strategy Nash equilibria for this game.

(b) This game also has a mixed-strategy Nash equilibrium. Find the probabilities the players use in this equilibrium, and provide an explanation for your answer.

(c) Keeping in mind Schelling's focal point idea from Chapter 5, what equilibrium do you think is the best prediction of how the game will be played? Explain.

Part B

(4) In class we discussed games with two players and we defined the concept of Nash equilibrium for these games. All of the ideas carry over to games with more than two players. Here we consider a game with three players, named 1, 2 and 3. To define the game we need to specify the sets of strategies available to each player; also, when each of the three players chooses a strategy, this gives a triple of strategies, and we need to specify the payoff each player receives from any possible triple of strategies played. Let's suppose that player 1's strategy set is $\{U, D\}$, players 2's strategy set is $\{L, R\}$ and player 3's strategy set is $\{l, r\}$.

One way to specify the payoffs would be to write down every possible triple of strategies, and the payoffs for each. A different but equivalent way to interpret triples of strategies, which makes it easier to specify the payoffs, is to imagine that player 3 chooses which of two distinct two-player games players 1 and 2 will play. In the payoff matrices for these games, the ones given in Figures 1 and 2, the first entry in each cell is the payoff to player 1, the second entry is the payoff to player 2 and the third entry is the payoff to player 3.

If 3 chooses l then the payoff matrix is:

		Player B	
		L	R
Player A	U	$4, 4, 4$	$0, 0, 1$
	D	$0, 2, 1$	$2, 1, 0$

Figure 1: Payoff Matrix l

If 3 chooses r then the payoff matrix is:

		Player B	
		L	R
Player A	U	$2, 0, 0$	$1, 1, 1$
	D	$1, 1, 1$	$2, 2, 2$

Figure 2: Payoff Matrix r

So, for example, if player 1 chooses U, player 2 chooses R and player 3 chooses r the payoffs are 1 for each player.

(a) First suppose the players all move simultaneously. That is, players 1 and 2 do not observe which game player 3 has selected until after they each chose a strategy. Find all of the (pure strategy) Nash equilibria for this game.

(b) Now suppose that player 3 gets to move first and that players 1 and 2 observe player 3's move before they decide how to play. That is, if player 3 chooses the strategy r then players 1 and 2 play the game defined by payoff matrix r and they both know that they are playing this game. Similarly, if player 3 chooses the strategy l then players 1 and 2 play the game defined by payoff matrix l and they both know that they are playing this game.

Let's also suppose that if players 1 and 2 play the game defined by payoff matrix r they play a (pure strategy) Nash equilibrium for that game; and similarly, if players 1 and 2 play the game defined by payoff matrix l they play a (pure strategy) Nash equilibrium for that game. Finally, let's suppose that player 3 understands that this is how players 1 and 2 will behave.

What do you expect player 3 to do and why? What triple of strategies would you expect to see played? Is this list of strategies a Nash equilibrium of the simultaneous move game between the three players?

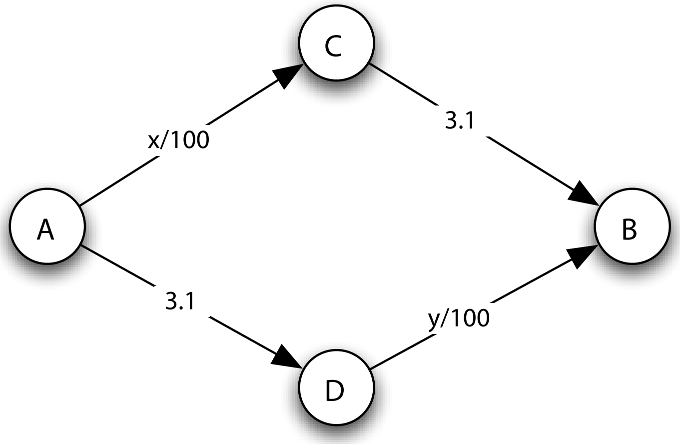

Figure 3: Traffic Network

(5) There are 300 cars which must travel from city A to city B. There are two possible routes that each car can take. The upper route through city C or the lower route through city D. Let x be the number of cars traveling on the edge AC and let y be the number of cars traveling on the edge DB. The directed graph in Figure 3 indicates that total travel time per car along the upper route is $(x/100) + 3.1$ if x cars use the upper route, and similarly the total travel time per car along the lower route is $3.1 + (y/100)$ if y cars take the lower route. Each driver wants to select a route to minimize his total travel time. The drivers make simultaneous choices.

(a) Find Nash equilibrium values of x and y.

(b) Now the government builds a new (one-way) road from city A to city B. The new route has a travel time of 5 per car regardless of the number of cars that use it. Draw the new network and label the edges with the cost-of-travel needed to move along the edge. The network should be a directed graph as all roads are one-way. Find a Nash equilibrium for the game played on the new network. What happens to total cost-of-travel (the sum of total travel times for the 300 cars) as a result of the availability of the new road?

(c) Now the government closes the direct route between city A and city B and builds a new one-way road which links city C to city D. This new road between C and D is very short and has a travel time of 0 regardless of the number of cars that use it. Draw the new network and label the edges with the cost-of-travel needed to move along the edge. The network should be a directed graph as all roads are one-way. Find a Nash equilibrium for the game played on the new network. What happens to total cost-of-travel as a result of the availability of the new road?

(d) The government is unhappy with the outcome in part (c) and decides to reopen the road directly linking city A and city B (the road that was built in part (b) and closed in part (c)). The AB road still has a travel time of 5 per car regardless of the number of cars that use it. The route between C and D that was constructed in part (c) remains open. Draw the new network and label the edges with the cost-of-travel needed to move along the edge. The network should be a directed graph as all roads are one-way. Find a Nash equilibrium for the game played on the new network. What happens to total cost-of-travel as a result of re-opening the direct route between A and B?